Distributed Remote Vector Gaussian Source Coding with Covariance Distortion Constraints

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***Abstract*—In this paper, we consider a distributed remote source coding problem, where a sequence of observations of source vectors is available at the encoder. The problem is to specify the optimal rate for encoding the observations subject to a covariance matrix distortion constraint and in the presence of side information at the decoder. For this problem, we derive lower and upper bounds on the rate-distortion function (RDF) for the Gaussian case, which in general do not coincide. We then provide some cases, where the RDF can be derived exactly. We also show that previous results on specific instances of this problem can be generalized using our results. We finally show that if the distortion measure is the mean squared error, or if it is replaced by a certain mutual information constraint, the optimal rate can be derived from our main result.**

I. Introduction

A. Notation and Problem Statement

We consider a stationary Gaussian source which generates independent vectors $\mathbf{x} \in \mathbb{R}^{n_x}$. A sequence of Gaussian vectors $\mathbf{y} \in \mathbb{R}^{n_y}$ which are measurements of the source is available at the encoder. Furthermore, a sequence of Gaussian vectors $\mathbf{z} \in \mathbb{R}^{n_z}$ is available at the decoder as side information. The problem is to specify the minimum rate for encoding $\mathbf{y}$ into a variable $\mathbf{u}$, so that the best estimation of the source from $\mathbf{u}$ and $\mathbf{z}$ at the decoder, denoted by $\hat{\mathbf{x}}$, satisfy a distortion constraint defined in form of a covariance matrix. This set-up is illustrated in Fig. 1.

We denote conditional and nonconditional covariance and cross-covariance matrices by symbol $\mathbf{\Sigma}$ followed by an appropriate subscript. We assume that all covariance matrices are of full rank. Matrices and vectors are denoted by boldface uppercase and lowercase letters, respectively. A diagonal matrix having the elements $\lambda_1, ..., \lambda_n$ on its main diagonal is denoted by $\text{diag}\{\lambda_i, i = 1, ..., n\}$. Markov chains are denoted by two-headed arrows; e.g. $\mathbf{y} \leftrightarrow \mathbf{x} \leftrightarrow \mathbf{z}$, and the trace operation is denoted by $\text{tr}(\cdot)$. We use $\mathbf{A} \succeq \mathbf{B}$ to show that $\mathbf{A} - \mathbf{B}$ is positive semidefinite. Finally, we make use of the following notations:

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$$(x)^+ \triangleq \max(x, 1), \tag{1}$$

$$(x)^- \triangleq \min(x, 1). \tag{2}$$

Using our notational convention, the problem described above can be formulated as specifying a rate-distortion function (RDF) $R(\mathbf{D})$, defined as:

$$R(\mathbf{D}) = \min_{\mathbf{u} \in \mathcal{U}} I(\mathbf{y}; \mathbf{u}|\mathbf{z}) \tag{3}$$

$$\text{subject to } E\left[(\mathbf{x} - \hat{\mathbf{x}})(\mathbf{x} - \hat{\mathbf{x}})^T\right] \preceq \mathbf{D}, \tag{4}$$

where $\mathbf{D}$ is a symmetric positive-definite matrix specifying the target distortion, $\hat{\mathbf{x}}$ is defined as:

$$\hat{\mathbf{x}} = E[\mathbf{x}|\mathbf{u}, \mathbf{z}], \tag{5}$$

and $\mathcal{U}$ is the set of random variables $\mathbf{u}$ satisfying $\mathbf{u} \leftrightarrow \mathbf{y} \leftrightarrow (\mathbf{x}, \mathbf{z})$.

Following [1], and for simplicity of derivations we write the Gaussian vectors $\mathbf{x}$ and $\mathbf{y}$ in terms of linear estimation from other Gaussian vectors and estimation errors as follows:

$$\mathbf{x} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u} + \mathbf{n_1}, \tag{6}$$

$$\mathbf{x} = \mathbf{C}\mathbf{y} + \mathbf{G}\mathbf{z} + \mathbf{n_2}, \tag{7}$$

$$\mathbf{y} = \mathbf{\Gamma}\mathbf{z} + \mathbf{n_3}, \tag{8}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{G}$ and $\mathbf{\Gamma}$ are the coefficients of linear estimation, depending only on the covariance and cross-covariance matrices of $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and $\mathbf{u}$, and $\mathbf{n}_i$, $i = 1, 2, 3$ are estimation errors with covariance matrices $\mathbf{\Sigma}_{\mathbf{n}_1}$, $\mathbf{\Sigma}_{\mathbf{n}_2} = \mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}}$, and $\mathbf{\Sigma}_{\mathbf{n}_3} = \mathbf{\Sigma}_{\mathbf{y}|\mathbf{z}}$, respectively. (See the Appendix in [4] for more details.)

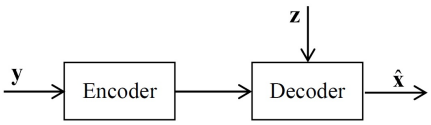

Fig. 1. Block diagram of the remote source coding problem

B. Applications

One possible application of the formulated problem is in wireless acoustic sensor networks which is a set of wireless microphones equipped with communication and signal processing units. The microphones are randomly distributed in an environment, sampling the sound field. The measurements made by each microphone should be delivered at a fusion center possibly via a sequence of transmissions through neighboring nodes. It is desirable to compute the local RDF at each node of such a network. This can be used for computing the network sum-rate, by which one can formulate a sum-rate minimization problem with a distortion constraint (e.g. distortion at the final destination). As suggested in [2], this can be used for optimal routing in the network. When sending a message from a node to a neighboring node, the measurement of the sound at the latter node can be used as side information, thus fitting our distributed source coding framework.

Another application is in relay networks as discussed in [1]. In this case, n_x, n_y, and n_z are the number of transmitter, relay, and receiver antennas.

C. Related Work

In the special case where $n_x = n_y = n_z = 1$, the RDF for the above-mentioned problem is given by [3]:

$$R(D) = \begin{cases} \frac{1}{2}\log\left(\frac{\Sigma_{x|z}-\Sigma_{x|yz}}{D-\Sigma_{x|yz}}\right), & \text{if } \Sigma_{x|yz} < D \leq \Sigma_{x|z} \\ 0, & \text{if } D > \Sigma_{x|z} \end{cases} \tag{9}$$

where D is the scalar distortion constraint, and $\Sigma_{x|z}$ and $\Sigma_{x|yz}$ are conditional variances of scalar random variable x.

For the vector case, the authors in [1] solved the problem for the mean-squared error distortion constraint; i.e. for problem (3) with the distortion constraint (4) replaced by:

$$\operatorname{tr}\left(E\left[\left(\mathbf{x}-\hat{\mathbf{x}}\right)\left(\mathbf{x}-\hat{\mathbf{x}}\right)^T\right]\right) \leq n_x D. \tag{10}$$

This is similar to a case where in (4), only the sum of the diagonal elements of the distortion and error matrices are of interest. Thus, in this particular case, the vector Gaussian problem can be treated as parallel scalar problems, leading to well-known water-filling interpretations. The RDF for this problem when $\operatorname{tr}(\mathbf{\Sigma_{x|yz}}) \leq n_x D \leq \operatorname{tr}(\mathbf{\Sigma_{x|z}})$ was shown in [1] to be:

$$R(D) = \frac{1}{2}\sum_{i=1}^{n_x}\log\left(\frac{\lambda_i}{\lambda}\right)^+, \tag{11}$$

where $\lambda_i;\ i = 1, 2, ..., n_x$ are the eigenvalues of $\mathbf{C\Sigma_{y|z}C}^T$ with $\mathbf{C}$ defined in (7), and λ satisfying the following constraint:

$$\sum_{i=1}^{n_x}\min\left(\lambda, \lambda_i\right) = n_x D - \operatorname{tr}\left(\mathbf{\Sigma_{x|yz}}\right). \tag{12}$$

Related to our problem is also another problem considered in [1], where the constraint (4) is replaced by a mutual information constraint defined as:

$$I\left(\mathbf{x};\mathbf{u}|\mathbf{z}\right) \geq R_I, \tag{13}$$

where R_I is a given rate. The rate-rate function for this problem for $0 \leq R_I \leq \frac{1}{2}\log\left(\frac{|\mathbf{\Sigma}_{x|\mathbf{z}}|}{|\mathbf{\Sigma}_{x|\mathbf{yz}}|}\right)$ is then given by:

$$R\left(R_I\right) = \frac{1}{2}\sum_{i=1}^{n_x}\log\left(\mu_i\left[\left(\frac{1-\mu_i}{1-\gamma}\right)^- - \left(1-\mu_i\right)\right]^{-1}\right), \tag{14}$$

where $\mu_i,\ i = 1, ..., n_x$ are the eigenvalues of $\mathbf{\Sigma}_{\mathbf{y|z}}^{1/2}\mathbf{C}^T\mathbf{\Sigma}_{\mathbf{x|z}}^{-1}\mathbf{C}\mathbf{\Sigma}_{\mathbf{y|z}}^{1/2}$, and $\gamma \in [0, 1)$ satisfies the following:

$$-\frac{1}{2}\sum_{i=1}^{n_x}\log\left(\frac{1-\mu_i}{1-\gamma}\right)^- = R_I. \tag{15}$$

In general, it is not straightforward to generalize the above results to the case of covariance matrix distortion constraints. Indeed, due to the matrix form of the distortion constraint, it does not appear as it is possible to reduce the problem to an equivalent problem of parallel scalar sources.

In [4], the RDF for (4) was recently found for the somewhat restrictive case where $n_x = n_y = n_z$, and $\mathbf{C}$ in (7) is invertible, and the distortion constraint satisfies $\mathbf{\Sigma_{x|yz}} \prec \mathbf{D} \preceq \mathbf{\Sigma_{x|z}}$. Under these assumptions, the RDF was shown to be:

$$R\left(\mathbf{D}\right) = \frac{1}{2}\log\left(\frac{\left|\mathbf{\Sigma}_{x|\mathbf{z}} - \mathbf{\Sigma}_{x|\mathbf{yz}}\right|}{\left|\mathbf{D} - \mathbf{\Sigma}_{x|\mathbf{yz}}\right|}\right). \tag{16}$$

Although under the above assumptions the problem is manageable to solve, it is a quite restricted case. In this paper, we consider the most general case with $\mathbf{\Sigma_{x|yz}} \prec \mathbf{D}$ and without the above assumptions, and establish a lower bound and an upper bound on the RDF, which in general do not coincide. Then we consider some special cases for which the two bounds coincide, giving the exact RDF. We will show that (9) and (11) could be derived as special cases of our results. In addition, in the special case that $\mathbf{y}$ and $\mathbf{z}$ are noisy versions of $\mathbf{x}$ with additive white noise, (14) could also be derived from our results. We will also generalize (16) to the case that no assumption is made on dimensions of vectors or invertibility of matrix $\mathbf{C}$.

The paper is organized as follows. Section II is dedicated to a brief presentation of some results from matrix algebra which will be used in our derivations. In Section III, we derive

the lower and upper bounds on the RDF for the problem formulated above. In Section IV, we will establish the link between our results and (9), (11), (14), and (16). The paper is concluded in Section V.

II. Simultaneous Diagonalization

The following theorem is a weakened variant of Theorem 8.3.1 in [5], and will be the basis for some of the derivations in this work:

Theorem 1. *For two symmetric positive definite $n\times n$ matrices $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$, there is a nonsigular matrix $\mathbf{S}$ so that:*

$$\mathbf{S}\mathbf{\Sigma}_1\mathbf{S}^T = \mathbf{I}_n, \tag{17}$$
$$\mathbf{S}\mathbf{\Sigma}_2\mathbf{S}^T = \mathbf{\Gamma}, \tag{18}$$

where $\mathbf{I}_n$ is the identity $n \times n$ matrix, and $\mathbf{\Gamma}$ is a positive-definite $n \times n$ diagonal matrix.

Let us denote the eigenvalue decomposition of $\mathbf{\Sigma}_1$ by:

$$\mathbf{\Sigma}_1 = \mathbf{U}^T\mathbf{\Lambda}\mathbf{U}. \tag{19}$$

We define the joint diagonalizer of $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ as:

$$\mathbf{V} = \mathbf{\Lambda}^{1/2}\mathbf{S}. \tag{20}$$

Using (20) and (17)–(18) we have:

$$\mathbf{V}\mathbf{\Sigma}_1\mathbf{V}^T = \mathbf{\Lambda}, \tag{21}$$
$$\mathbf{V}\mathbf{\Sigma}_2\mathbf{V}^T = \mathbf{\Lambda}', \tag{22}$$

where $\mathbf{\Lambda}'$ is defined as $\mathbf{\Lambda}' = \mathbf{\Lambda}\mathbf{\Gamma}$. Note that the diagonal elements in $\mathbf{\Lambda}'$ are not necessarily the eigenvalues of $\mathbf{\Sigma}_2$. However, if $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ commute, it is possible to find a joint eigenvalue decomposition for the two matrices, so that the matrix $\mathbf{V}$ in (21) and (22) is orthogonal, and $\mathbf{\Lambda}'$ consists of the eigenvalues of $\mathbf{\Sigma}_2$.

We will also make use of the following theorem (see [5], Theorem 8.4.9):

Theorem 2. *Consider two positive semidefinite matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$, with eigenvalues $\lambda_1, ..., \lambda_n$ and $\mu_1, ..., \mu_n$, respectively, which are sorted in order of magnitude. If $\mathbf{Q}_1 \succeq \mathbf{Q}_2$, then $\lambda_i \geq \mu_i$, for $i = 1, ..., n$.*

III. Main Results

Let us define the matrices $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ as:

$$\mathbf{\Sigma}_1 = \mathbf{\Sigma}_{\mathbf{x}|\mathbf{z}} - \mathbf{\Sigma}_{\mathbf{x}|\mathbf{y}\mathbf{z}}, \tag{23}$$
$$\mathbf{\Sigma}_2 = \mathbf{D} - \mathbf{\Sigma}_{\mathbf{x}|\mathbf{y}\mathbf{z}}, \tag{24}$$

and denote their simultaneous diagonalization by $\mathbf{\Lambda} = \text{diag}\{\lambda_i, i = 1, ..., n_x\}$ and $\mathbf{\Lambda}' = \text{diag}\{\lambda'_i, i = 1, ..., n_x\}$, respectively. The eigenvalue decomposition of $\mathbf{\Sigma}_1$ is defined in (19). We also denote by $\lambda_{(i)}$ and $\lambda'_{(i)}$, $i = 1, ..., n_x$, the sorted-by-magnitude versions of λ_i and λ'_i, $i = 1, ..., n_x$, respectively. The following theorem is the main result of this paper:

Theorem 3. *The RDF formulated in (3)–(4) is bounded as follows:*

$$\frac{1}{2}\sum_{i=1}^{n_x}\log\left(\frac{\lambda_{(i)}}{\lambda'_{(i)}}\right)^+ \leq R(\mathbf{D}) \leq \frac{1}{2}\sum_{i=1}^{n_x}\log\left(\frac{\lambda_i}{\lambda'_i}\right)^+ \tag{25}$$

Proof: The proof follows from the results of the next two subsections. We first propose a scheme which achieves the upper bound. Then we prove that the RDF can be lower-bounded as in (25). ■

A. Upper Bound

We will show that the upper bound in (25) is achievable by the following scheme:

$$\mathbf{u} = \mathbf{U}\mathbf{C}\mathbf{y} + \nu, \tag{26}$$

where the covariance matrix of the coding noise ν is defined as:

$$\mathbf{\Sigma}_\nu = \mathbf{U}\mathbf{V}^{-1}\text{diag}\left\{\frac{\lambda_i \min(\lambda_i, \lambda'_i)}{\lambda_i - \min(\lambda_i, \lambda'_i)}, i = 1, ..., n_x\right\}\mathbf{V}^{-T}\mathbf{U}^T. \tag{27}$$

Proof: First notice that using (7) we can write:

$$\mathbf{C}\mathbf{\Sigma}_{\mathbf{y}|\mathbf{z}}\mathbf{C}^T = \mathbf{\Sigma}_{\mathbf{x}|\mathbf{z}} - \mathbf{\Sigma}_{\mathbf{x}|\mathbf{y}\mathbf{z}}. \tag{28}$$

Starting from $I(\mathbf{y}; \mathbf{u}|\mathbf{z}) = h(\mathbf{u}|\mathbf{z}) - h(\mathbf{u}|\mathbf{y}, \mathbf{z})$ and using (26), (8), and (28), it is straightforward to show that:

$$I(\mathbf{y}; \mathbf{u}|\mathbf{z}) = \frac{1}{2}\log\left(\frac{|\mathbf{\Lambda} + \mathbf{\Sigma}_\nu|}{|\mathbf{\Sigma}_\nu|}\right). \tag{29}$$

Noting that

$$\mathbf{U}\mathbf{V}^{-1}\mathbf{\Lambda}\mathbf{V}^{-T}\mathbf{U}^T = \mathbf{\Lambda}, \tag{30}$$

and substituting (27) in (29) yields:

$$\begin{aligned} I(\mathbf{y}; \mathbf{u}|\mathbf{z}) &= \frac{1}{2}\sum_{i=1}^{n_x}\log\left(\frac{\lambda_i}{\min(\lambda_i, \lambda'_i)}\right) \\ &= \frac{1}{2}\sum_{i=1}^{n_x}\log\left(\frac{\lambda_i}{\lambda'_i}\right)^+. \end{aligned} \tag{31}$$

Now we will show that using the coding scheme (26) the reconstruction error at the decoder satisfies the distortion constraint (4). First notice that form (6) it follows that $\mathbf{\Sigma}_{xu|z} = \mathbf{B}\mathbf{\Sigma}_{u|z}$, or:

$$\mathbf{B} = \mathbf{\Sigma}_{xu|z}{\mathbf{\Sigma}_{u|z}}^{-1}. \tag{32}$$

From (26) and (28) we have:

$$\mathbf{\Sigma}_{u|z} = \mathbf{\Lambda} + \mathbf{\Sigma}_{\nu}. \tag{33}$$

Also:

$$\begin{aligned}\mathbf{\Sigma}_{xu|z} &= \mathbf{\Sigma}_{xy|z}\mathbf{C}^T\mathbf{U}^T &(34)\\ &= \mathbf{C}\mathbf{\Sigma}_{y|z}\mathbf{C}^T\mathbf{U}^T &(35)\\ &= \left(\mathbf{\Sigma}_{x|z} - \mathbf{\Sigma}_{x|yz}\right)\mathbf{U}^T, &(36)\end{aligned}$$

where (34), (35) and (36) follow from (26), (7) and (28), respectively. The covariance matrix of the reconstruction error can then be written as:

$$\begin{aligned}&E\left[(\mathbf{x}-\hat{\mathbf{x}})(\mathbf{x}-\hat{\mathbf{x}})^T\right] = \mathbf{\Sigma}_{\mathbf{n}_1} &(37)\\ &= \mathbf{\Sigma}_{\mathbf{x}|\mathbf{z}} - \mathbf{B}\mathbf{\Sigma}_{\mathbf{u}|\mathbf{z}}\mathbf{B}^T &(38)\\ &= \mathbf{\Sigma}_{\mathbf{x}|\mathbf{z}} - \mathbf{\Sigma}_{\mathbf{xu}|\mathbf{z}}{\mathbf{\Sigma}_{\mathbf{u}|\mathbf{z}}}^{-1}{\mathbf{\Sigma}_{\mathbf{xu}|\mathbf{z}}}^T &(39)\\ &= \mathbf{\Sigma}_{\mathbf{x}|\mathbf{z}} - \left(\mathbf{\Sigma}_{\mathbf{x}|\mathbf{z}} - \mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}}\right)\mathbf{U}^T(\mathbf{\Lambda}+\mathbf{\Sigma}_{\nu})^{-1}\mathbf{U}\left(\mathbf{\Sigma}_{\mathbf{x}|\mathbf{z}} - \mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}}\right), &(40)\end{aligned}$$

where (37) and (38) follow from (5) and (6), (39) is result of substituting (32) in (38), and (40) follows from (33) and (36). Using (21), (23), (30) and (27), we can rewrite (40) as follows:

$$\begin{aligned}&E\left[(\mathbf{x}-\hat{\mathbf{x}})(\mathbf{x}-\hat{\mathbf{x}})^T\right]\\ &= \mathbf{\Sigma}_{\mathbf{x}|\mathbf{z}} - \mathbf{V}^{-1}\mathrm{diag}\left\{\lambda_i - \min\left(\lambda_i, \lambda_i'\right), i = 1, ..., n_x\right\}\mathbf{V}^{-T} &(41)\\ &= \mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}} + \mathbf{V}^{-1}\mathrm{diag}\left\{\min\left(\lambda_i, \lambda_i'\right), i = 1, ..., n_x\right\}\mathbf{V}^{-T}. &(42)\end{aligned}$$

From (24) and (22) we have:

$$\mathbf{D} = \mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}} + \mathbf{V}^{-1}\mathrm{diag}\left\{\lambda_i', i = 1, ..., n_x\right\}\mathbf{V}^{-T}. \tag{43}$$

Comparing (43) and (42), it is clear that $E\left[(\mathbf{x}-\hat{\mathbf{x}})(\mathbf{x}-\hat{\mathbf{x}})^T\right] \preceq \mathbf{D}$.

■

B. Lower Bound

Let us denote the quantized encoded sequence by $\mathbf{w}$, and define $\mathbf{s}$ as $\mathbf{s} \triangleq \mathbf{C}\mathbf{y} + \mathbf{G}\mathbf{z}$. Then from (7) we have $\mathbf{n}_2 = \mathbf{x} - \mathbf{s}$. The reconstruction error can be written as:

$$\mathbf{x} - \hat{\mathbf{x}} = \mathbf{n}_2 + \upsilon, \tag{44}$$

where $\mathbf{n}_2$ is the error resulting from irrelevant information in $\mathbf{y}$ and $\mathbf{z}$ (remote source coding), and υ is the error due to rate constraints. From (44) and the fact that $\mathbf{\Sigma}_{\mathbf{n}_2} = \mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}}$ we have:

$$\mathrm{cov}\left(\mathbf{s}-\hat{\mathbf{x}}\right) = \mathbf{\Sigma}_{\upsilon} = E\left[(\mathbf{x}-\hat{\mathbf{x}})(\mathbf{x}-\hat{\mathbf{x}})^T\right] - \mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}}. \tag{45}$$

Starting from (45) we can write the following chain of inequalities:

$$\begin{aligned}&\left|E\left[(\mathbf{x}-\hat{\mathbf{x}})(\mathbf{x}-\hat{\mathbf{x}})^T\right] - \mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}}\right| = \left|\mathrm{cov}\left(\mathbf{s}-\hat{\mathbf{x}}\right)\right| &(46)\\ &\geq \frac{1}{(2\pi e)^{n_x}}\exp\left[2h\left(\mathbf{s}-\hat{\mathbf{x}}\right)\right] &(47)\\ &\geq \frac{1}{(2\pi e)^{n_x}}\exp\left[2h\left(\mathbf{s}-\hat{\mathbf{x}}|\hat{\mathbf{x}}\right)\right] &(48)\\ &\geq \frac{1}{(2\pi e)^{n_x}}\exp\left[2h\left(\mathbf{s}|\hat{\mathbf{x}}, \mathbf{z}\right)\right] &(49)\\ &\geq \frac{1}{(2\pi e)^{n_x}}\exp\left[2h\left(\mathbf{s}|\mathbf{z}\right) - 2I\left(\mathbf{s}; \mathbf{w}|\mathbf{z}\right)\right] &(50)\\ &= \left|\mathbf{\Sigma}_{\mathbf{x}|\mathbf{z}} - \mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}}\right|\exp\left[-2I\left(\mathbf{s}; \mathbf{w}|\mathbf{z}\right)\right]\\ &\geq \left|\mathbf{\Sigma}_{\mathbf{x}|\mathbf{z}} - \mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}}\right|\exp\left[-2H\left(\mathbf{w}\right)\right] &(51)\\ &\geq \left|\mathbf{\Sigma}_{\mathbf{x}|\mathbf{z}} - \mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}}\right|\exp\left(-2R\right), &(52)\end{aligned}$$

where (47) is because Gaussian distribution maximizes the differential entropy, (48) is because conditioning reduces the entropy, (49) is result of the fact that $h\left(\mathbf{s}-\hat{\mathbf{x}}|\hat{\mathbf{x}}\right) = h\left(\mathbf{s}|\hat{\mathbf{x}}\right)$ and conditioning reduces the entropy, (50) can be obtained from the following chain:

$$\begin{aligned}h\left(\mathbf{s}|\hat{\mathbf{x}}, \mathbf{z}\right) &= h\left(\mathbf{s}\right) - I\left(\mathbf{s}; \hat{\mathbf{x}}, \mathbf{z}\right)\\ &\geq h\left(\mathbf{s}\right) - I\left(\mathbf{s}; \hat{\mathbf{x}}, \mathbf{z}, \mathbf{w}\right)\\ &= h\left(\mathbf{s}\right) - I\left(\mathbf{s}; \mathbf{z}, \mathbf{w}\right)\\ &= h\left(\mathbf{s}\right) - \left\{I\left(\mathbf{s}; \mathbf{z}\right) + I\left(\mathbf{s}; \mathbf{w}|\mathbf{z}\right)\right\}\\ &= h\left(\mathbf{s}|\mathbf{z}\right) - I\left(\mathbf{s}; \mathbf{w}|\mathbf{z}\right), &(53)\end{aligned}$$

and (51) follows from the following inequalities:

$$\begin{aligned}I\left(\mathbf{s}; \mathbf{w}|\mathbf{z}\right) &= H(\mathbf{w}|\mathbf{z}) - H(\mathbf{w}|\mathbf{s}, \mathbf{z})\\ &\leq H(\mathbf{w}|\mathbf{z})\\ &\leq H(\mathbf{w}), &(54)\end{aligned}$$

and thus $-I\left(\mathbf{s}; \mathbf{w}|\mathbf{z}\right) \geq -H(\mathbf{w})$. From (52) we have:

$$\begin{aligned}R &\geq \frac{1}{2}\log\left(\frac{\left|\mathbf{\Sigma}_{\mathbf{x}|\mathbf{z}} - \mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}}\right|}{\left|E\left[(\mathbf{x}-\hat{\mathbf{x}})(\mathbf{x}-\hat{\mathbf{x}})^T\right] - \mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}}\right|}\right) &(55)\\ &= \frac{1}{2}\log\left(\frac{|\mathbf{\Lambda}|}{\left|\mathbf{V}\left(E\left[(\mathbf{x}-\hat{\mathbf{x}})(\mathbf{x}-\hat{\mathbf{x}})^T\right] - \mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}}\right)\mathbf{V}^T\right|}\right) &(56)\end{aligned}$$

where (56) follows from (21) and (23). Let us denote the eigenvalues of $\mathbf{V}\left(E\left[(\mathbf{x}-\hat{\mathbf{x}})(\mathbf{x}-\hat{\mathbf{x}})^T\right] - \mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}}\right)\mathbf{V}^T$ sorted in order of magnitude by $\mu_{(1)}, ..., \mu_{(n_x)}$. From the fact

that $E\left[\left(\mathbf{x}-\hat{\mathbf{x}}\right)\left(\mathbf{x}-\hat{\mathbf{x}}\right)^T\right] \preceq \mathbf{\Sigma}_{\mathbf{x}|\mathbf{z}}$ [1] and Theorem 2 we have $\mu_{(i)} \leq \lambda_{(i)}$, $i = 1, ..., n_x$. From the distortion constraint we have $\mathbf{V}\left(E\left[\left(\mathbf{x}-\hat{\mathbf{x}}\right)\left(\mathbf{x}-\hat{\mathbf{x}}\right)^T\right] - \mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}}\right)\mathbf{V}^T \preceq \mathbf{V}\left(\mathbf{D}-\mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}}\right)\mathbf{V}^T$, which when combined with (22), (24) and Theorem 2 yields $\mu_{(i)} \leq \lambda'_{(i)}$, $i = 1, ..., n_x$. Therefore we can write:

$$\mu_{(i)} \leq \min\left\{\lambda'_{(i)}, \lambda_{(i)}\right\},\ i = 1, ..., n_x. \tag{57}$$

From (56) we have:

$$\begin{aligned} R \quad &\geq \frac{1}{2}\log\left(\frac{\prod_{i=1}^{n_x}\lambda_{(i)}}{\prod_{i=1}^{n_x}\mu_{(i)}}\right) \\ &\geq \frac{1}{2}\sum_{i=1}^{n_x}\log\left(\frac{\lambda_{(i)}}{\min\left\{\lambda'_{(i)}, \lambda_{(i)}\right\}}\right) \qquad (58) \\ &= \frac{1}{2}\sum_{i=1}^{n_x}\log\left(\frac{\lambda_{(i)}}{\lambda'_{(i)}}\right)^+, \qquad (59) \end{aligned}$$

where (58) follows from (57). The lower bound is established by (59).

IV. Special Cases

In this section, we create a link between (25) and (9), (11), (14) and (16). We start from the following corollary:

Corollary 1. *For $\mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}} \prec \mathbf{D} \preceq \mathbf{\Sigma}_{\mathbf{x}|\mathbf{z}}$ the RDF (3)–(4) is given by:*

$$R\left(\mathbf{D}\right) = \frac{1}{2}\log\left(\frac{\left|\mathbf{\Sigma}_{x|\mathbf{z}} - \mathbf{\Sigma}_{x|\mathbf{yz}}\right|}{\left|\mathbf{D} - \mathbf{\Sigma}_{x|\mathbf{yz}}\right|}\right). \tag{60}$$

Proof: We will show that in this special case the lower and upper bounds coincide to (60). First note that from the assumption, the distortion constraint and (55) we have:

$$R \geq \frac{1}{2}\log\left(\frac{\left|\mathbf{\Sigma}_{\mathbf{x}|\mathbf{z}} - \mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}}\right|}{\left|\mathbf{D} - \mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}}\right|}\right)$$

which proves the lower bound. From the assumption we can write (27) as:

$$\begin{aligned} \mathbf{\Sigma}_\nu &= \mathbf{U}\mathbf{V}^{-1}\mathrm{diag}\left\{\frac{\lambda_i\lambda'_i}{\lambda_i - \lambda'_i}, i = 1, ..., n_x\right\}\mathbf{V}^{-T}\mathbf{U}^T \qquad (61) \\ &= \mathbf{U}\mathbf{V}^{-1}\mathbf{\Lambda}\left(\mathbf{\Lambda} - \mathbf{\Lambda}'\right)^{-1}\mathbf{\Lambda}'\mathbf{V}^{-T}\mathbf{U}^T \\ &= \mathbf{U}\left(\mathbf{\Sigma}_{x|\mathbf{z}} - \mathbf{\Sigma}_{x|\mathbf{yz}}\right)\left(\mathbf{\Sigma}_{x|\mathbf{z}} - \mathbf{D}\right)^{-1}\left(\mathbf{D} - \mathbf{\Sigma}_{x|\mathbf{yz}}\right)\mathbf{U}^T, \qquad (62) \end{aligned}$$

where (61) follows from the fact that $\lambda'_i = \min\left(\lambda_i, \lambda'_i\right)$, and (62) follows from (21)–(24). Substituting (62) in (29), one can show that $I(\mathbf{y}; \mathbf{u}|\mathbf{z})$ is equal to (60). Following the same lines of argument as in Section III-A, one can show that the reconstruction error at the decoder is exactly the same as the target distortion. This completes the proof.

■

Note that Corollary 1 is a generalization of (16). Also note that (9) immediately follows from (60) by setting $n_x = n_y = n_z = 1$.

Let us define a subset $\mathcal{D}_1$ of the set of covariance distortion constraints $\mathcal{D}$ as all the covariance distortion constraints $\mathbf{D}$ for which $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ in (23) and (24) commute. Similarly, we define the subset $\mathcal{D}_2$ of $\mathcal{D}$ as the set of all covariance distortion constraints $\mathbf{D}$ which commute with $\mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}}$. In the sequel, we will provide two propositions which relate our results to (11) and (14). The proofs are left out due to space limitations.

Proposition 1. *Minimization of (60) over the set of all covariance matrix distortion constraints $\mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}} \prec \mathbf{D} \preceq \mathbf{\Sigma}_{\mathbf{x}|\mathbf{z}}$ in $\mathcal{D}_1$ which satisfy the mean-squared error constraint (10) yields (11) and (12).*

Proposition 2. *If $\mathbf{y} = \mathbf{x} + \mathbf{n}_1$ and $\mathbf{z} = \mathbf{x} + \mathbf{n}_2$ where the noise terms $\mathbf{n}_1$ and $\mathbf{n}_2$ are white, mutually independent and independent of $\mathbf{x}$, then minimization of (60) over the set of all covariance matrix distortion constraints $\mathbf{\Sigma}_{\mathbf{x}|\mathbf{yz}} \prec \mathbf{D} \preceq \mathbf{\Sigma}_{\mathbf{x}|\mathbf{z}}$ in $\mathcal{D}_2$ which satisfy $|\mathbf{D}| \leq e^{-2R_I}|\mathbf{\Sigma}_{\mathbf{x}|\mathbf{z}}|$ yields (14) and (15).*

V. Conclusions

We upper- and lower-bounded the rate-distortion function for the vector Gaussian remote source coding problem with side information at the decoder and covariance matrix distortion constraints. We further studied some special cases where the exact rate-distortion function can be derived. We showed that several results from existing works can be derived and generalized using these special cases. Future work includes the derivation of the exact rate-distortion function in the general case and also application of the results to the problem of source coding in wireless acoustic sensor networks.

[1] Note that $\mathbf{\Sigma}_{\mathbf{x}|\mathbf{z}}$ is the covariance of the reconstruction error for zero rate, therefore the reconstruction error cannot be larger than that. However, it does not mean that the distortion constraint has to be restricted to $\mathbf{D} \preceq \mathbf{\Sigma}_{\mathbf{x}|\mathbf{z}}$.

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